INTEGRABILITY OF FUNCTIONS REPRESENTED BY TRIGONOMETRIC SERIES

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CERTIFICATE

This is to certify that ***Mr. Zubair Khan*** has undertaken a special study of ***"Integrability Of Functions Represented By Trigonometric Series"*** under my supervision. His work is suitable for submission for the award of the degree of Master of Philosophy in Mathematics.

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PREFACE

Suppose that a periodic function f is associated with a trigonometric cosine series

$$\frac{1}{2}a_o + \sum_{n=1}^{\infty} a_n \cos nx,$$

under various circumstances we may want to suppose that f is integrable and that a_n are its Fourier Coefficients or that the series converges to the function. There are two questions :

(a) if ψ is a given positive function, and f belongs to a specified class of functions, what hypotheses on $\{a_n\}$ are equivalent to $f\psi \in \mathbf{L}$?

(b) if $\{\mu_n\}$ is a given sequence of positive numbers, and $\{a_n\}$ belongs to a given class of sequences, what hypotheses on f are equivalent to $\sum \mu_n |a_n| < \infty$?

Several mathematicians have studied these types of problems what we now term as integrability problems.

When $0 < \gamma < 1$, a function that behaves near 0 like $x^{-\gamma}$ has a cosine series whose coefficients behave at infinity like $n^{\gamma-1}$, and conversely. This fact dominates most of the theory of integrability of trigonometric series, so that we are concerned with conditions for the convergence of $\sum |a_n| n^{\gamma-1}$ or of $\int |f(x)| x^{-\gamma} dx$.

The main object of this dissertation is to discuss certain important generalizations of well known theorems on the integrability of functions represented by trigonometric series and power series.

Chapter–I deals with the definitions of certain basic concepts which play an important role in the theory of trigonometric series.

Chapter–II deals with the necessary and sufficient condition for f(x) to be integrable in the sense of Lebesgue.

Let us consider the trigonometric cosine series

$$f(x) = \frac{1}{2} a_o + \sum_{n=1}^{\infty} a_n \cos nx,$$

where the coefficients a_n decreases to zero monotonically. It is well known that under this condition f(x) is not integrable in L $(0, \pi)$. Now the question arises, what is the necessary and sufficient condition for f(x) to be integrable in the sense of Lebesgue ?

The sufficiency part of this question has been examined by several mathematicians like Young [16], Kolmogorove [10] etc. Thus, for example, Sidon [14] proved that if

$$a_n \underset{o}{\downarrow} \text{ and } \sum_{n=1}^{\infty} \frac{a_n}{n} < \infty,$$

then $f(x) \in L(o, \pi)$.

However the converse of his assertions is not true. Recently Rees and Stanojevic [11] considered a different type of cosine series and proved the necessary and sufficient conditions.

Chapter–III deals with the pth power integrability of functions represented by the trigonometric series. For example, Hardy and Littlewood proved that if $a_n \underset{o}{\downarrow}$ and $1 < p < \infty$, then $f \in L^p(o, \pi)$ if , and only if,

$$\sum_{n=1}^{\infty} n^{p-2} a_n^p < \infty.$$

This theorem also holds for sine series.

Chapter–IV deals with the integrability of functions represented by the power series. For example, Khan [9] proved the following theorem which reduces to the theorems of Askey [1], Askey and Boas [2] and Heywood [8] :

Let $F(x) = \sum_{k=o}^{\infty} a_k x^k, a_k \geq o, \quad o \leq x < 1,$

$S_n = \sum_{k=o}^{n} a_k$ and $\gamma < 1$. Then for $o < p < \infty$,

$$\int_o^1 (1-x)^{-\gamma} [F(x)]^p \, dx < \infty$$

if, and only if,

$$\sum_{n=1}^{\infty} n^{\gamma-2} S_n^p < \infty.$$

CONTENTS

CHAPTER - I

CHAPTER – I

(Preliminaries)

1.1 Trigonometric Series and Fourier Series :

A trigonometric series is a series of the form

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \qquad (1.1)$$

where a_n and b_n ($n = 0, 1, 2, \ldots..$) are independent of x and are known as the coefficients of the series.

If the series (1.1) converges for all x in $-\infty < x < +\infty$, then it represents a function of period 2π. The problem of representing a given function by a series of the form (1.1) was encountered by Fourier in a problem of conducting of heat.

Now we try to find the formula for the coefficients a_n, b_n in terms of the given function. Let the function f(x) is the sum of the trigonometric series (1.1) and the series converges uniformly on $[-\pi, \pi]$ then by multiplying

$$f(x) = \frac{1}{2} a_o + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

by cos mx or by sin mx, where m is a positive integer and integrating it between the limit - π to π and using the following formulas,

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = o, \; m \neq n,$$

$$\int_{-\pi}^{\pi} \sin mx \; \sin nx \, dx = o, \; m \neq n,$$

$$\int_{-\pi}^{\pi} \cos mx \sin nx \, dx = o, \; m \neq n \;\text{ and }\; m = n,$$

$$\int_{-\pi}^{\pi} \cos^2 mx \, dx = \int_{-\pi}^{\pi} \sin^2 mx \, dx = \pi .$$

we obtain the result

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \; (n = 0, 1, 2, ...) \qquad (1.2)$$

and $$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \; (n = 1, 2, 3, ...) \qquad (1.3)$$

The above formulae are called Fourier formulae (The above formulae were already known to Euler, but Fourier began to use them systematically and therefore they are called Fourier formulae).

Suppose that we are given a Lebesgue integrable function f(x) in [-π, π], then the integral (1.2) and (1.3) exist and the numbers a_n, b_n defined by them are called Fourier coefficients of f(x). The trigonometric series of the form (1.1) with these coefficients is called the Fourier Series of f(x) and we write

$$f(x) \sim \frac{1}{2} a_o + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

or $$\sigma(f) = \frac{1}{2} a_o + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

The sign ~ indicate that starting from f(x) and using Fourier formulae the series has been formed . Every integrable function f(x) defined on [- π, π] has its Fourier series but there are cases when the trigonometric series is given by its coefficients but we do not know whether it is a Fourier series of a function or not. This is an interesting but difficult problem of theory of trigonometric series.

1.2 Complex Form of Trigonometric Series and Fourier Series :

By Euler's formula

$$e^{ix} = \cos x + i \sin x,$$

we have

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \ \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

On putting these in (1.1) and then by supposing $c_o = a_o/2$,

$$c_n = \frac{a_n - ib_n}{2}, \ c_{-n} = \frac{a_n + ib_n}{2},$$

we have

$$\frac{1}{2}a_o + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \sum_{n=-\infty}^{n=+\infty} c_n e^{inx} .$$

The series

$$\sum_{n=-\infty}^{n=+\infty} c_n e^{inx}$$

is called the complex form of the trigonometric series (1.1) and the numbers c_n and c_{-n} are the conjugate complex numbers, that is,

$$\underline{c}_n = \bar{c}_n .$$

If the series representing f(x) is given in complex form

$$\sum_{n=-\infty}^{n=+\infty} c_n e^{inx} ,$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx \quad (n = o, \pm 1, \pm 2, \ldots\ldots)$$

are called the complex Fourier coefficient of the function f(x) and the series is called the complex Fourier series of the function f(x) and we write

$$f(x) \sim \sum_{n=-\infty}^{n=+\infty} c_n e^{inx}$$

1.3 Fourier – Stieltjes Series :

Let F(x) be a function of bounded variation defined on $[o, 2\pi]$. Let us consider the series $\sum_{n=-\infty}^{n=+\infty} c_n e^{inx}$ with coefficients given by

$$c_n = \frac{1}{2\pi} \int_o^{2\pi} e^{-inx}\, dF \quad (n = o, \pm 1, \pm 2, \ldots\ldots)$$

where the integral in the above formula is Riemann – Stieltjes integral. The numbers c_n are called the Fourier – Stieltjes coefficients of F or the Fourier coefficients of dF and we write

$$dF(x) \sim \sum_{n=-\infty}^{n=+\infty} c_n e^{inx}$$

and call the series as the Fourier – Stieltjes series of F or the Fourier series of dF. We can also write the Fourier series of dF in the form

$$\frac{1}{2} a_o + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) ,$$

where

$$a_n = \frac{1}{\pi} \int_o^{2\pi} \cos nx \, dF ,$$

and $$b_n = \frac{1}{\pi} \int_o^{2\pi} \sin nx \, dF .$$

1.4 Fourier Series for Even and Odd Functions :

If the function f(x) is even, that is, f (-x) = f (x), then

$$\sigma(f) = \frac{1}{2} a_o + \sum_{n=1}^{\infty} a_n \cos nx ,$$

where

$$a_n = \frac{2}{\pi} \int_o^{\pi} f(x) \cos nx \, dx .$$

and if the function f(x) is odd, that is, f(-x) = - f(x), then

$$\sigma(f) = \sum_{n=1}^{\infty} b_n \sin nx ,$$

where

$$b_n = \frac{2}{\pi} \int_o^{\pi} f(x) \sin nx \, dx .$$

1.5 Properties of Fourier Coefficients :

(a) Fourier coefficients of any integrable function tends to zero.

(b) If f(x) is a function of bounded variation on [o, 2π] and V is its total variation on [o, 2π], then

$$| a_n | \leq \frac{V}{2n} ,$$

$$\text{and} \quad | b_n | \leq \frac{V}{2n} .$$

Thus, if f(x) is of bounded variation,

$$\left.\begin{array}{l} a_n \\ \\ b_n \end{array}\right\} = O\left(\frac{1}{n}\right)$$

1.6 Formal Operations on Fourier Series :

(a) Addition and subtraction of Fourier series : If

$$f(x) \sim \sum_{n=-\infty}^{n=+\infty} c_n e^{inx}$$

and $\quad g(x) \sim \sum_{n=-\infty}^{n=+\infty} \gamma_n e^{inx}$, then

$$f(x) \pm g(x) \sim \sum_{n=-\infty}^{n=+\infty} (c_n \pm \gamma_n) e^{inx},$$

where $c_n \pm \gamma_n = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} [f(x) \pm g(x)]\, e^{-inx}\, dx$.

(b) Multiplication by a constant : If

$$f(x) \sim \sum_{n=-\infty}^{n=+\infty} c_n e^{inx}, \text{ then}$$

$$kf(x) \sim \sum_{n=-\infty}^{n=+\infty} kc_n e^{inx} ,$$

where k is any constant.

(c) Fourier series for $f(x + \alpha)$: If α is any constant and

$$f(x) \sim \sum_{n=-\infty}^{n=+\infty} c_n e^{inx} , \text{ then}$$

$$f(x + \alpha) \sim \sum_{n=-\infty}^{n=+\infty} c_n e^{in(x+\alpha)} .$$

(d) Differentiation of Fourier series : If $f(x)$ is absolutely continuous on $[a, b]$ and if

$$f(x) \sim \sum_{n=-\infty}^{n=+\infty} c_n e^{inx} , \text{ then}$$

$$f'(x) \sim \sum_{n=-\infty}^{n=+\infty} in\, c_n e^{inx} .$$

If $$f(x) \sim \frac{1}{2} a_o + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \text{ then}$$

$$f'(x) \sim \sum_{n=1}^{\infty} n(b_n \cos nx - a_n \sin nx) .$$

(e) Integration of Fourier Series :

Any Fourier series, whether convergent or not, may be integrated term – by – term between any limits, with in which the function is integrable, that is, the sum of the integrals of the separate terms is the integral of the function of which the series is the Fourier series.

Let a_n, b_n be the Fourier coefficients of f(x), and let

$$g(x) = \int_o^x \left(f(t) - \frac{1}{2} a_o \right) dt,$$

then g(x) is periodic, continuous and of bounded variation. Hence g can be expanded in a Fourier series, say

$$g(x) = \frac{1}{2} A_o + \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx)$$

convergent for all values of x, where

$$A_n = -\frac{b_n}{n}, \quad B_n = \frac{a_n}{n}$$

Hence $$g(x) = \frac{A_o}{2} + \sum_{n=1}^{\infty} \frac{a_n \sin nx - b_n \cos nx}{n}.$$

On putting x = o, we have

$$g(o) = \frac{A_o}{2} + \sum_{n=1}^{\infty} - \frac{b_n}{n}$$

$$\text{or} \quad \frac{1}{2} A_o = \sum_{n=1}^{\infty} \frac{b_n}{n}.$$

Thus

$$g(x) = \sum_{n=1}^{\infty} \frac{a_n \sin nx + b_n (1 - \cos nx)}{n}$$

Hence for any α and β

$$\int_{\alpha}^{\beta} f(x)\, dx = \left[\frac{1}{2} a_o x \right]_{\alpha}^{\beta} + \sum_{n=1}^{\infty} \left[\frac{- b_n \cos nx + a_n \sin nx}{n} \right]_{\alpha}^{\beta},$$

that is, Fourier series (even divergent) can be integrated term – by – term in any interval.

An interesting particular case is that for any Fourier series, the series $\sum_{n=1}^{\infty} \frac{b_n}{n}$ converges. Thus, we can write convergent trigonometric series which are not Fourier series. For example, let us consider the series

$$\sum_{n=2}^{\infty} \frac{\sin nx}{\log n} ,$$

which converges at every point, but is not a Fourier series of its sum because the series $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ diverges.

(f) A Fourier series may be multiplied by any function of bounded variation and integrated term – by – term between any finite limits.

1.7 Integrability of Trigonometric Series :

Suppose that a periodic function f is associated with a trigonometric cosine series

$$\frac{1}{2} a_o + \sum_{n=1}^{\infty} a_n \cos nx ,$$

under various circumstances we may want to suppose that f is integrable, or that the series converges to the function. Now there are two questions :

(a) if ψ is a given positive function, and f belongs to a specified class of functions, what hypotheses on $\{a_n\}$ are equivalent to $f\psi \in L$?

(b) if $\{\mu_n\}$ is a given sequence of positive numbers, and $\{a_n\}$ belongs to a given class of sequences, what hypotheses on f are equivalent to $\sum \mu_n |a_n| < \infty$?

Several mathematicians have studied these types of problems what we now term as integrability problems.

CHAPTER - II

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Integrability of Certain Cosine Sums

2.1 A sequence $\{a_n\}$ is said to be monotonic decreasing if $a_{n+1} \leq a_n$, $n = 1, 2, 3, \ldots$. It is said to be a null sequence if $a_n \rightarrow o$ as $n \rightarrow \infty$.

The idea of decreasing null sequence was generalized in the form of a quasi-monotone sequence by shah [12] and Szasz [15] in the following manner.

A sequence $\{a_n\}$ of positive numbers is said to be quasi-monotone if, and only if, $n^{-\beta} a_n \downarrow o$ for some $\beta \geq o$ or equivalently, if $n \Delta a_n \geq - \alpha a_n$ for some constant $\alpha \geq o$ where $\Delta a_n = a_n - a_{n+1}$.

It is clear that if $\{a_n\}$ is a monotonic decreasing null sequence then it is also quasi-monotonic.

However, the converse need not be true. The quasi-monotonic sequences are known to share many of the important properties of decreasing sequences. For example, Olivier's classical theorem, which states that if $\{a_n\}$ is a positive monotonic decreasing sequence and $\sum_{n=1}^{\infty} a_n < \infty$, then $n\, a_n \to o$ as $n \to \infty$, was extended for quasi-monotonic sequences by Szasz [15], Cauchy's condensation test for convergence and also a number of results about trigonometric series have been found to be true for quasi – monotonic sequences. Shah [13] proved the following theorem concerning the integrability of trigonometric series for quasi – monotonic sequences which extends the theorem of Boas [4] for monotonic null sequences.

Theorem : Let $\{a_n\}$ be quasi – monotonic. If $o < \gamma < 1$, then $\sum_{n=1}^{\infty} n^{\gamma-1} a_n$ is convergent if, and only if, $x^{-\gamma} f(x) \in L\,(o, \pi)$, where $f(x) = \sum_{n=1}^{\infty} a_n \cos nx$.

A sequence $\{a_n\}$ is called convex, if $\Delta^2 a_n \geq o$,

where $$\Delta^2 a_n = \Delta a_n - \Delta a_{n+1}$$

and $\qquad \Delta a_n = a_n - a_{n+1}$.

2.2 Let us consider the trigonometric cosine series

$$f(x) = \frac{1}{2} a_o + \sum_{n=1}^{\infty} a_n \cos n x , \qquad \textbf{(2.2.1)}$$

where the coefficients a_n decreases to zero monotonically. This series converges to a function f(x) everywhere except possibly at x = o [3]. It is clear from the following theorem that monotonicity does not ensure the L – integrability of f(x).

Theorem A [17]. There is a series (2.2.1) with coefficients monotonically decreasing to zero and its sum f(x) not integrable L .

In order to ensure L – integrability of f(x) Young [16] using an additional condition proved the following theorem :

Theorem B. If $a_n \underset{o}{\downarrow}$ and $\Delta^2 a_n \geq o$, then f(x) is L – integrable on (o, π). Theorem B was improved by Kolmogorov [10] in the following manner :

Theorem C. If $a_n = o\,(1)$ and $\sum_{n=1}^{\infty} (n+1)\,|\Delta^2 a_n| < \infty$, then

$f(x) \in L\,(o, \pi)$.

Sidon [14] used another condition to obtain the integrability of f(x). His result is as follows :

Theorem D. Let $a_n \underset{o}{\downarrow}$. If $\sum_{n=1}^{\infty} \frac{a_n}{n} < \infty$, then $f(x) \in L\,(o, \pi)$.

The condition $\sum_{n=1}^{\infty} \frac{a_n}{n} < \infty$ in the above theorem is not necessary as can be seen from the following example :

$$f(x) = \sum_{n=2}^{\infty} \frac{\cos nx}{\log n},$$

where $a_n = \frac{1}{\log n}$ decreases to zero monotonically and $\Delta^2 a_n \geq o$, so by theorem B, f(x) is L – integrable, but $\sum_{n=2}^{\infty} \frac{1}{n \log n} = \infty$.

Now the question arises, what is the necessary and sufficient condition for f(x) to be integrable ? This problem is still unsolved. However, if we assume some additional condition on f then Goes and Goes [6] proved the following theorem :

Theorem E. If $a_n \underset{o}{\downarrow}$, then $f(x) \in L\,(o, \pi)$ if, and only if, a_n are the Fourier - Stieltjes coefficients of a function of bounded variation.

In 1973, for a different types of cosine sums, Rees and Stanojevic [11] investigated this problem for L – Class and proved the following theorem involving necessary and sufficient conditions.

Theorem F. Let $b_k = \frac{a_k}{k} \underset{o}{\downarrow}$. Then

$$g\,(x) = \lim_{n \to \infty} \sum_{k=1}^{n} \left[\frac{b_k}{2} + \left(\sum_{j=k}^{n} b_j \right) \cos kx \right]$$

exists for $x \in (o, \pi)$ and $g(x) \in L\,(o, \pi]$ if, and only if,

$$\sum_{n=1}^{\infty} b_n < \infty .$$

Proof. Let

$$S_n(x) = \frac{1}{2}\sum_{k=1}^{n} b_k + \sum_{k=1}^{n}\left(\sum_{j=k}^{n} b_j\right)\cos kx .$$

On applying partial summation to the second term of the above, we get

$$S_n(x) = \frac{1}{2}\sum_{k=1}^{n} b_k + \sum_{k=1}^{n} b_k\left(D_k(x) - \frac{1}{2}\right)$$

$$= \frac{1}{2}\sum_{k=1}^{n} b_k + \sum_{k=1}^{n} b_k D_k(x) - \frac{1}{2}\sum_{k=1}^{n} b_k$$

$$= \sum_{k=1}^{n} b_k D_k(x),$$

where $D_k(x) = \frac{1}{2} + \cos x + \cos 2x + \ldots. + \cos kx$

$$= \frac{\sin\left(k + \frac{1}{2}\right)x}{2\sin\frac{x}{2}}$$

is Dirichlet Kernel.

Applying partial summation again, we get

$$S_n(x) = \sum_{k=1}^{n-1} (k + 1) F_k(x) (b_k - b_{k+1})$$

$$+ b_n (n + 1) F_n(x) - \frac{1}{2} b_1 ,$$

where $F_k(x)$ is Fejer's Kernel, that is to say

$$F_k(x) = \frac{1}{k + 1} \sum_{n=o}^{k} D_n(x)$$

$$= \frac{\sin^2 \left(\frac{k + 1}{2} \right) x}{2 (k + 1) \sin^2 \frac{x}{2}}$$

Now, from the assumption that $b_n \to o$, we have

$$\lim_{n \to \infty} (n + 1) b_n F_n(x) = o$$

Since $F_n(x) = O\left(\frac{1}{(n + 1) x^2} \right)$, $x \neq o$, we have

$$\sum_{k=1}^{n-1} (k + 1)(b_k - b_{k+1}) F_k(x) \leq \frac{C}{x^2} \sum_{k=1}^{n-1} (b_k - b_{k+1}),$$

where C is some positive constant.

Now, $b_n \downarrow_{0}$ then $\sum_{k=1}^{\infty} |b_k - b_{k+1}| < \infty$ and so it follows that

$\sum_{k=1}^{\infty} (k+1)(b_k - b_{k+1}) F_k(x)$ converges. Therefore

$$g(x) = \lim_{n \to \infty} S_n(x) = \sum_{k=1}^{\infty} (k+1)(b_k - b_{k+1}) F_k(x) - \frac{1}{2} b_1 \text{ for}$$

$x \in (o, \pi)$.

Now, let

$$g(x) = \sum_{k=1}^{\infty} (k+1)(b_k - b_{k+1}) F_k(x) - \frac{1}{2} b_1$$

which is a convergent series with non – negative terms, so that

$$\int_o^{\pi} g(x)\,dx = \sum_{k=1}^{\infty} (k+1)(b_k - b_{k+1}) \int_o^{\pi} F_k(x)\,dx - \frac{1}{2} b_1 \int_o^{\pi} dx$$

$$= \frac{\pi}{2} \sum_{k=1}^{\infty} (k+1)(b_k - b_{k+1}) - \frac{\pi}{2} b_1$$

$$= \frac{\pi}{2} \left[\sum_{k=1}^{\infty} (k+1)(b_k - b_{k+1}) - b_1 \right]$$

Now, by partial summation, we have

$$\sum_{k=1}^{n} b_k = \sum_{k=1}^{n-1} (k+1)\Delta b_k + (n+1) b_n - b_1$$

Since $\quad b_n \underset{o}{\downarrow}$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $n\, b_n \to o$.

Therefore by taking the limit, we get

$$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} (k+1)\Delta b_k - b_1$$

Hence $\displaystyle\int_o^{\pi} g(x)\,dx = \frac{\pi}{2} \sum_{k=1}^{\infty} b_k$

$$< \infty \qquad \text{iff } \sum_{k=1}^{\infty} b_k < \infty .$$

This completes the proof of the theorem.

Remark. It seems that the above theorem can be improved by replacing "$b_n \underset{o}{\downarrow}$" by quasi – monotonicity of $\{b_n\}$.

CHAPTER - III

CHAPTER – III

P^{th} Power Integrability of Trigonometric Series

3.1 Let us consider the trigonometric series

$$\frac{1}{2} a_o + \sum_{n=1}^{\infty} a_n \cos nx, \qquad \textbf{(3.1)}$$

where the coefficients a_n decreases to zero monotonically, then in any interval $o < \delta \leq x \leq 2\pi - \delta$, the series (3.1) converges uniformly to a function f(x) [3]. But on approaching the point x = o, f(x) can increase with out bound and even be non – integrable in (o, π) [17]. It is known that for $o < \gamma < 1$, a function that behaves near o like $x^{-\gamma}$ has a cosine series whose coefficients behave at infinity like $n^{\gamma - 1}$ and conversely. This fact dominates most of the theory of integrability of trigonometric series.

In 1952, Boas [4] proved the following theorem which is of the above kind.

Theorem A. If $a_n \underset{O}{\downarrow}$ and $f(x) = \sum_{n=1}^{\infty} a_n \cos nx$, then for $o < \gamma < 1$, $x^{-\gamma} f(x) \in L(o, \pi)$ if, and only if, $\sum_{n=1}^{\infty} n^{\gamma-1} a_n < \infty$.

If $\gamma = o$ as observed in Chapter – II, only the sufficiency part of the theorem is known to be true.

In this Chapter, we shall discuss p^{th} power of integrability of trigonometric series. In this direction there is a classical result of Hardy and Littlewood [7]. Their result is as follows :

Theorem B. If $a_n \underset{O}{\downarrow}$ and $1 < p < \infty$, then $f(x) \in L^p(o, \pi)$ if, and only if,

$$\sum_{n=1}^{\infty} n^{p-2} a_n^p < \infty.$$

If $p = 1$, it is clear from our previous remark that although sufficiency part is true, the necessity part breaks down.

Theorem B was generalized by Chen [5] who established a necessary and sufficient conditions for the integrability of $x^{-\gamma}\{f(x)\}^p$ for $p > 1$. His results are as follows :

Theorem C. If $a_n \underset{o}{\downarrow}$ then for $p > 1$ and $o < \gamma < 1$, $x^{-\gamma}\{f(x)\}^p \in L(o, \pi)$ if, and only if,

$$\sum_{n=1}^{\infty} n^{\gamma + p - 2} a_n^p < \infty.$$

Theorem D. If $a_n \underset{o}{\downarrow}$ then (p^{th} power integrability) $x^{-\gamma} f(x) \in L^p (o, \pi)$, $p > 1$ and $\frac{1-p}{p} < \gamma < \frac{1}{p}$, if and only if,

$$\sum_{n=1}^{\infty} n^{p\gamma + p - 2} a_n^p < \infty.$$

3.2 We would like to give the proof of theorem B (Hardy and Littlewood) which is of fundamental nature. We use the following lemmas for the proof of the theorem.

Lemma 1. If $f(x) \in L^p [-\pi, \pi]$ and $\phi(x) = \int_o^x |f(t)|\, dt$,

then

$$\int_0^\pi \left[\frac{\phi(x)}{x} \right]^p dx \le C \int_0^\pi |f(x)|^p dx , \; p > 1,$$

where C is dependent only on p.

Lemma 2. Let $C > 1$, $p > 1$ and suppose $A_n = \sum_{k=1}^{n} a_k$. Then

$$\sum_{n=1}^{\infty} n^{-C} A_n^p \le k \sum_{n=1}^{\infty} n^{-C} (na_n)^p .$$

Proof of Theorem B.

Necessary Condition : Let $f(x) \in L^p(o, \pi)$, that is, $\int_o^\pi |f(x)|^p dx < \infty$. Then the series $\sum_{n=1}^{\infty} a_n \cos nx$ is its Fourier series and hence the series can be integrated term – by – term.

$$\text{Let} \quad F(x) = \int_0^x f(t)\, dt$$

$$= \int_0^x \sum_{n=1}^{\infty} a_n \cos nt \, dt$$

$$= \sum_{n=1}^{\infty} a_n \int_0^x \cos nt \, dt$$

$$= \sum_{n=1}^{\infty} \frac{a_n}{n} \sin nx$$

$$\therefore F(\pi / k) = \sum_{n=1}^{\infty} \frac{a_n}{n} \sin n \frac{\pi}{k}$$

Since $\quad \sin n \frac{\pi}{k} = - \sin (n + k) \frac{\pi}{k} = \sin (n + 2k) \frac{\pi}{k} = \ldots\ldots..$, then

$$F(\pi / k) = \sum_{n=1}^{k-1} \left(\frac{a_n}{n} - \frac{a_{n+k}}{n + k} + \frac{a_{n+2k}}{n + 2k} - \ldots \right) \sin n \frac{\pi}{k}$$

$$\geq \sum_{n=1}^{k-1} \left(\frac{a_n}{n} - \frac{a_{n+k}}{n + k} \right) \sin n \frac{\pi}{k}$$

If $\frac{k}{4} < n \leq \frac{k}{2}$, then

$$\frac{k}{4} \cdot \frac{\pi}{k} < n \cdot \frac{\pi}{k} \leq \frac{k}{2} \cdot \frac{\pi}{k}$$

or $$\frac{\pi}{4} < n \frac{\pi}{k} \leq \frac{\pi}{2}$$

$$\therefore \quad \sin n \frac{\pi}{k} > \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

or $$\sin n \frac{\pi}{k} > \frac{1}{\sqrt{2}}$$

Also, since $a_n \downarrow$,

$$a_{n+k} \leq a_n$$

or $$-a_{n+k} \geq -a_n$$

or $$-\frac{a_{n+k}}{n+k} \geq -\frac{a_n}{n+k}$$

or $$\frac{a_n}{n} - \frac{a_{n+k}}{n+k} \geq \frac{a_n}{n} - \frac{a_n}{n+k}$$

$$= a_n \left(\frac{1}{n} - \frac{1}{n+k} \right)$$

$$\geq \frac{a_n}{n}$$

Hence $F(\pi/k) \geq B \sum_{n=1}^{k-1} \frac{a_n}{n}$

$$\geq B \frac{a_{k-1}}{k-1} \sum_{n=1}^{k-1} 1$$

$$= B\, a_{k-1}$$

$$\geq B\, a_k$$

or $\qquad a_k \leq C\, F(\pi/k)$

Now $\qquad a_n^p \leq C_1 F^p(\pi/n)$

or $\qquad n^{p-2} a_n^p \leq C_1 n^{p-2} F^p(\pi/n)$

or $\quad \sum_{n=1}^{\infty} n^{p-2} a_n^p \leq C_1 \sum_{n=1}^{\infty} n^{p-2} F^p(\pi/n)$

$$= C_1 \sum_{n=1}^{\infty} n^{p-2} \left(\int_0^{\pi/n} f(t)\, dt \right)^p$$

$$\leq C_1 \sum_{n=1}^{\infty} n^{p-2} \left(\int_0^{\pi/n} | f(t) |\, dt \right)^p$$

$$= C_1 \sum_{n=1}^{\infty} n^{p-2} (\phi(\pi/n))^p ,$$

where $\phi(x) = \int_0^x |f(t)|\,dt$

$$\text{Now } \int_{\pi/n}^{\pi/n-1} \left(\frac{\phi(x)}{x}\right)^p dx \geq (\phi(\pi/n))^p \int_{\pi/n}^{\pi/n-1} \frac{1}{x^p}\,dx$$

$$= C_2\,(\phi(\pi/n))^p\; n^{p-2}$$

Thus from above, we have

$$\sum_{n=1}^{\infty} n^{p-2} a_n^p \leq C_3 \sum_{n=2}^{\infty} \int_{\pi/n}^{\pi/n-2} \left(\frac{\phi(x)}{x}\right)^p dx$$

$$= C_4 \int_0^{\pi} \left(\frac{\phi(x)}{x}\right)^p dx$$

$$= C_5 \int_0^{\pi} |f(x)|^p\, dx\,, \text{ by lemma 1.}$$

$$< \infty$$

Sufficient Condition : Let

$$\sum_{n=1}^{\infty} n^{p-2} a_n^p < \infty.$$

$$\text{Let} \qquad f(x) = \sum_{k=1}^{\infty} a_k \cos kx$$

$$= \sum_{k=1}^{n} a_k \cos kx + \sum_{k=n+1}^{\infty} a_k \cos kx$$

On applying Abel's transformation to the second term,

we have

$$f(x) = \sum_{k=1}^{n} a_k \cos kx + \sum_{k=n}^{\infty} \Delta a_k D_k(x) - a_n D_n(x)$$

$$\text{Thus } |f(x)| \le \sum_{k=1}^{n} a_k + \sum_{k=n}^{\infty} |\Delta a_k| |D_k(x)| + a_n |D_n(x)|$$

$$= A_n + \sum_{k=n}^{\infty} |\Delta a_k| |D_k(x)| + a_n |D_n(x)|,$$

where $D_n(x) = \frac{1}{2} + \cos x + \cos 2x + \ldots\ldots + \cos nx$,

and $A_n = \sum_{k=1}^{n} a_k$.

Since $|D_n(x)| \le \frac{\pi}{x}$ for $o < x < \pi$, and $a_n \downarrow o$.

$$|f(x)| \le A_n + \frac{\pi}{x} a_n \quad \text{for} \quad o < x < \pi$$

and therefore

$$|f(x)| \le C A_n \quad \text{for } \frac{\pi}{n+1} \le x \le \frac{\pi}{n}$$

$$\text{Now } \int_0^{\pi} |f(x)|^p \, dx = \sum_{n=1}^{\infty} \int_{\pi/n+1}^{\pi/n} |f(x)|^p \, dx$$

$$\leq C_1 \sum_{n=1}^{\infty} \int_{\pi/n+1}^{\pi/n} A_n^p \, dx$$

$$= C_1 \sum_{n=1}^{\infty} A_n^p \int_{\pi/n+1}^{\pi/n} dx$$

$$\leq C_2 \sum_{n=1}^{\infty} A_n^p \, n^{-2}$$

$$\leq C_3 \sum_{n=1}^{\infty} n^{p-2} a_n^p, \text{ by Lemma 2.}$$

$$< \infty$$

This completes the proof of theorem B.

CHAPTER - IV

CHAPTER – IV

Integrability of Power Series

4.1 Given a sequence $\{a_n\}$ of real numbers, the series

$$\sum_{n=o}^{\infty} a_n x^n$$

is called a power series (or the power series generated by $\{a_n\}$) and a_n is called the coefficient of x^n. Thus the power series is a function of x provided it converges for some or all x. Of course, it converges for x = o. The convergence for other values of x depends on the choice of coefficients $\{a_n\}$. Given any sequence $\{a_n\}$ one of the following holds for its power series :

(a) the power series converges for all $x \in R$,

(b) the power series converges only for x = o ,

(c) the power series converges for all x in some interval (symmetrical about o) and diverges outside the interval. The interval may be open, half-open or closed.

The above remarks are consequences of the following theorem.

Theorem : For the power series

$$\sum_{n=o}^{\infty} a_n x^n ,$$

let

$$\beta = \lim_{n \to \infty} \sup |a_n|^{1/n}$$

and

$$R = \frac{1}{\beta} ,$$

where $R = +\infty$ if $\beta = o$ and $R = o$ if $\beta = +\infty$. Then

(i) the power series converges for $|x| < R$,

(ii) the power series diverges for $|x| > R$,

where R is called the radius of convergence for the power series.

4.2. In 1955 Heywood [8] proved the following theorem concerning the integrability of power series.

Theorem A : Let

$$F(x) = \sum_{k=o}^{\infty} a_k x^k , \quad a_k \geq o , \; o \leq x < 1 .$$

Then for $\gamma < 1$,

$$(1-x)^{-\gamma} F(x) \in L(o, 1)$$

if, and only if,

$$\sum_{n=1}^{\infty} n^{\gamma-1} a_n < \infty .$$

Askey [1] and Askey and Boas [2] proved the following theorems respectively concerning L^p behaviour of power series with positive coefficients.

Theorem B : Let

$$F(x) = \sum_{k=o}^{\infty} a_k x^k , \quad a_k \geq o , \; o \leq x < 1 .$$

Then for $1 \leq p < \infty$,

$$\int_o^1 [F(x)]^p \, dx < \infty$$

if, and only if,

$$\sum_{n=1}^{\infty} n^{-2} \left(\sum_{k=o}^{n} a_k \right)^p < \infty .$$

Theorem C : Let

$$F(x) = \sum_{k=o}^{\infty} a_k x^k , \quad a_k \geq o , \; o \leq x < 1 .$$

Then for $o < p < 1$,

$$\int_o^1 [F(x)]^p \, dx < \infty$$

if, and only if,

$$\sum_{n=1}^{\infty} n^{-2} \left(\sum_{k=o}^{n} a_k \right)^p < \infty .$$

In 1977 Khan [9] proved the following theorem which generalizes all the above theorems.

Theorem D : Let

$$F(x) = \sum_{k=o}^{\infty} a_k x^k \ , \ a_k \geq o \ , \ o \leq x < 1 \, ,$$

$$S_n = \sum_{k=o}^{n} a_k \ , \text{ and } \gamma < 1 \, .$$

Then for $o < p < \infty$,

$$\int_o^1 (1 - x)^{-\gamma} \left[F(x) \right]^p dx < \infty$$

if, and only if,

$$\sum_{n=1}^{\infty} n^{\gamma - 2} S_n^p < \infty \, .$$

If $\gamma = o$ then theorem D reduces to theorem B and C and for $p = 1$ it reduces to theorem A of Heywood on account of the following result :

Let $a_n \geq o$ and $\gamma < 1$. Then

$$\sum_{n=1}^{\infty} n^{\gamma - 1} a_n < \infty$$

if, and only if,

$$\sum_{n=1}^{\infty} n^{\gamma - 2} S_n < \infty ,$$

where

$$S_n = \sum_{k=o}^{n} a_k .$$

Proof of theorem D : Let $o < p < \infty$.

Necessary Condition : Let $1 - x = y$. Then by virtue of the fact that

$$\left(1 - \frac{1}{n} \right)^n$$

is an increasing sequence, we have

for

$$\frac{1}{n+1} \leq y \leq \frac{1}{n} , n \geq 2$$

$$F(1 - y) = \sum_{k=o}^{\infty} a_k (1 - y)^k$$

$$\geq \sum_{k=o}^{n} a_k (1 - y)^k$$

$$\geq \sum_{k=o}^{n} a_k (1 - \frac{1}{n})^k$$

$$\geq \left(1 - \frac{1}{n}\right)^n \sum_{k=o}^{n} a_k$$

$$\geq \frac{1}{4} \sum_{k=o}^{n} a_k$$

$$= \frac{1}{4} S_n .$$

Thus,

$$F(1 - y) \geq A S_n \quad \text{for} \quad \frac{1}{n+1} \leq y \leq \frac{1}{n} \quad (n = 2, 3, \ldots.),$$

where A is a positive constant not necessarily the same at each occurrence.

Now

$$\sum_{n=1}^{\infty} n^{\gamma-2} S_n^p \leq A \sum_{n=1}^{\infty} \int_n^{n+1} t^{\gamma-2} S_{[t]}^p \, dt$$

$$= A \int_1^{\infty} t^{\gamma-2} S_{[t]}^p \, dt$$

$$= A \int_o^1 u^{-\gamma} S_{\left[\frac{1}{u}\right]}^p \, du$$

$$= A \sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} u^{-\gamma} S_n^p \, du$$

$$= A \int_{\frac{1}{2}}^{1} u^{-\gamma} S_1^p \, du + A \sum_{n=2}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} u^{-\gamma} S_n^p \, du$$

$$\leq A + A \int_o^{\frac{1}{2}} y^{-\gamma} [\, F(1 - y)]^p \, dy$$

$$\leq A + A \int_0^1 y^{-\gamma} [F(1 - y)]^p \, dy$$

$$= A + A \int_0^1 (1 - x)^{-\gamma} [F(x)]^p \, dx$$

$$< \infty .$$

Sufficient Condition :

Case (i) Let $1 \leq p < \infty$. On writing $(1 - x) = y$, we have

$$\int_0^1 (1 - x)^{-\gamma} [F(x)]^p \, dx$$

$$= \sum_{n=1}^{\infty} \int_{1-\frac{1}{n}}^{1-\frac{1}{n+1}} (1 - x)^{-\gamma} [F(x)]^p \, dx$$

$$= \sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} y^{-\gamma} [F(1 - y)]^p \, dy$$

$$= \sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} y^{-\gamma} \left[\sum_{k=0}^{\infty} a_k (1 - y)^k \right]^p dy$$

$$\leq \sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} y^{-\gamma} \left[\sum_{k=0}^{\infty} a_k \left(1 - \frac{1}{n+1}\right)^k \right]^p dy$$

$$\leq A \sum_{n=1}^{\infty} n^{\gamma - 2} \left[\sum_{k=0}^{\infty} a_k \left(1 - \frac{1}{n+1}\right)^k \right]^p$$

$$= A \sum_{n=1}^{\infty} n^{\gamma - 2} \left[\sum_{j=0}^{\infty} \sum_{k=nj}^{n(j+1)} a_k \left(1 - \frac{1}{n+1}\right)^k \right]^p$$

$$\leq A \sum_{n=1}^{\infty} n^{\gamma - 2} \left[\sum_{j=0}^{\infty} \left(1 - \frac{1}{n+1}\right)^{nj} \sum_{k=nj}^{n(j+1)} a_k \right]^p$$

$$\equiv K_1 \text{ , say.}$$

since

$$\frac{1}{2} \geq (1 - \frac{1}{n+1})^n \geq (1 - \frac{1}{n+2})^{n+1} \quad \text{for} \quad n = 1, 2, 3, \ldots ,$$

we have

$$\begin{aligned}
K_1 &\leq A \sum_{n=1}^{\infty} n^{\gamma-2} \left[\sum_{j=0}^{\infty} 2^{-j} \sum_{k=0}^{n(j+1)} a_k \right]^p \\
&= A \sum_{n=1}^{\infty} n^{\gamma-2} \left[\sum_{i=1}^{\infty} 2^{-i} \sum_{k=0}^{ni} a_k \right]^p \\
&= A \sum_{n=1}^{\infty} n^{\gamma-2} \left[\sum_{i=1}^{\infty} 2^{-i} S_{ni} \right]^p \\
&= A \sum_{n=1}^{\infty} n^{\gamma-2} \left[\sum_{i=1}^{\infty} 2^{-\frac{i}{2}} 2^{-\frac{i}{2}} S_{ni} \right]^p \\
&\leq A \sum_{n=1}^{\infty} n^{\gamma-2} \sum_{i=1}^{\infty} 2^{-\frac{ip}{2}} S_{ni}^p \\
&\leq A \sum_{i=1}^{\infty} 2^{-\frac{ip}{2}} \sum_{n=1}^{\infty} n^{\gamma-2} S_{ni}^p \\
&= A \sum_{i=1}^{\infty} 2^{-\frac{ip}{2}} i^{-\gamma+2} \sum_{n=1}^{\infty} (in)^{\gamma-2} S_{ni}^p \\
&\leq A \sum_{n=1}^{\infty} n^{\gamma-2} S_n^p \sum_{i=1}^{\infty} 2^{-\frac{ip}{2}} i^{-\gamma+2} \\
&\leq A \sum_{n=1}^{\infty} n^{\gamma-2} S_n^p
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_0^1 (1 - x)^{-\gamma} [F(x)]^p \, dx &\leq A \sum_{n=1}^{\infty} n^{\gamma-2} S_n^p \\
&< \infty .
\end{aligned}$$

Case (ii) Let $o < p < 1$.

Let $(1 - x) = y$.

Then

$$\int_o^1 (1-x)^{-\gamma} [F(x)]^p \, dx$$

$$= \sum_{n=1}^{\infty} \int_{1-\frac{1}{n}}^{1-\frac{1}{n+1}} (1-x)^{-\gamma} [F(x)]^p \, dx$$

$$= \sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} y^{-\gamma} [F(1-y)]^p \, dy$$

$$= \sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} y^{-\gamma} \left[\sum_{k=o}^{\infty} a_k (1-y)^k \right] dy$$

$$\leq \sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} y^{-\gamma} \left[\sum_{k=o}^{\infty} a_k \left(1 - \frac{1}{n+1}\right)^k \right]^p dy$$

$$\leq A \sum_{n=1}^{\infty} n^{\gamma-2} \left[\sum_{k=o}^{\infty} a_k \left(1 - \frac{1}{n+1}\right)^k \right]^p$$

$$= A \sum_{n=1}^{\infty} n^{\gamma-2} \left[\sum_{k=o}^{n} a_k \left(1 - \frac{1}{n+1}\right)^k + \sum_{k=n+1}^{\infty} a_k \left(1 - \frac{1}{n+1}\right)^k \right]^p$$

$$\leq A \sum_{n=1}^{\infty} n^{\gamma-2} \left[\sum_{k=o}^{n} a_k \left(1 - \frac{1}{n+1}\right)^k \right]^p$$

$$+ A \sum_{n=1}^{\infty} n^{\gamma-2} \left[\sum_{k=n}^{\infty} a_k \left(1 - \frac{1}{n+1}\right)^k \right]^p$$

$$\leq A \sum_{n=1}^{\infty} n^{\gamma-2} S_n^p + A \sum_{n=1}^{\infty} n^{\gamma-2} \left[\sum_{k=n}^{\infty} a_k \left(1 - \frac{1}{n+1}\right)^k \right]^p$$

$$= J_1 + J_2 \text{, say}.$$

$$\text{Now } J_2 = A \sum_{n=1}^{\infty} n^{\gamma-2} \left[\sum_{k=n}^{\infty} a_k \left(1 - \frac{1}{n+1}\right)^k \right]^p$$

$$= A \sum_{n=1}^{\infty} n^{\gamma-2} \left[\sum_{j=1}^{\infty} \sum_{k=nj}^{n(j+1)} a_k \left(1 - \frac{1}{n+1}\right)^k \right]^p$$

$$\leq A \sum_{n=1}^{\infty} n^{\gamma-2} \left[\sum_{j=1}^{\infty} \left(1 - \frac{1}{n+1}\right)^{nj} \sum_{k=nj}^{n(j+1)} a_k \right]^p$$

$$\leq A \sum_{n=1}^{\infty} n^{\gamma-2} \left[\sum_{j=1}^{\infty} 2^{-j} \sum_{k=nj}^{n(j+1)} a_k \right]^p$$

$$\leq A \sum_{n=1}^{\infty} n^{\gamma-2} \sum_{j=1}^{\infty} 2^{-jp} \left[\sum_{k=nj}^{n(j+1)} a_k \right]^p$$

$$\leq A \sum_{n=1}^{\infty} n^{\gamma-2} \sum_{j=1}^{\infty} 2^{-jp} \left[\sum_{k=o}^{n(j+1)} a_k \right]^p$$

$$\leq A \sum_{n=1}^{\infty} n^{\gamma-2} \sum_{i=1}^{\infty} 2^{-ip} S_{ni}^p$$

$$= A \sum_{i=1}^{\infty} 2^{-ip} \sum_{n=1}^{\infty} n^{\gamma-2} S_{ni}^p$$

$$= A \sum_{i=1}^{\infty} 2^{-ip} i^{-\gamma+2} \sum_{n=1}^{\infty} (in)^{\gamma-2} S_{ni}^p$$

$$\leq A \sum_{n=1}^{\infty} n^{\gamma-2} S_n^p \sum_{i=1}^{\infty} 2^{-ip} i^{-\gamma+2}$$

$$\leq A \sum_{n=1}^{\infty} n^{\gamma-2} S_n^p$$

Thus,

$$\int_0^1 (1-x)^{-\gamma} [F(x)]^p \, dx \leq A \sum_{n=1}^{\infty} n^{\gamma-2} S_n^p$$

$$< \infty .$$

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